Week 15 - Monday

COMP 2230

Last time

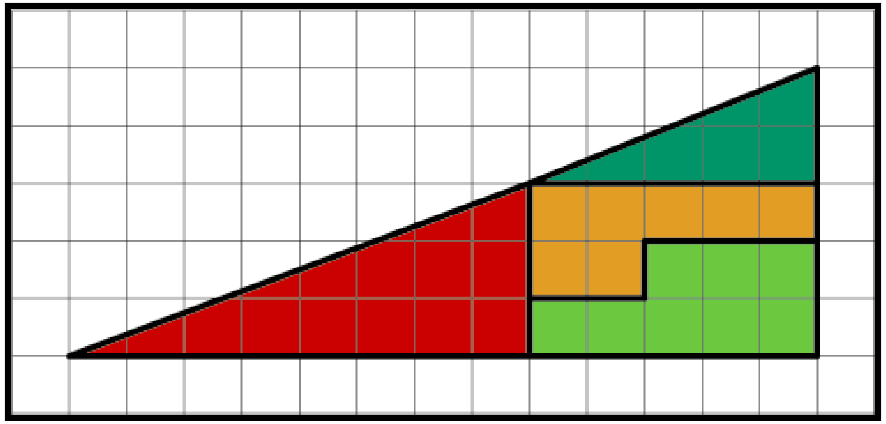
- Reviewed first third of the course

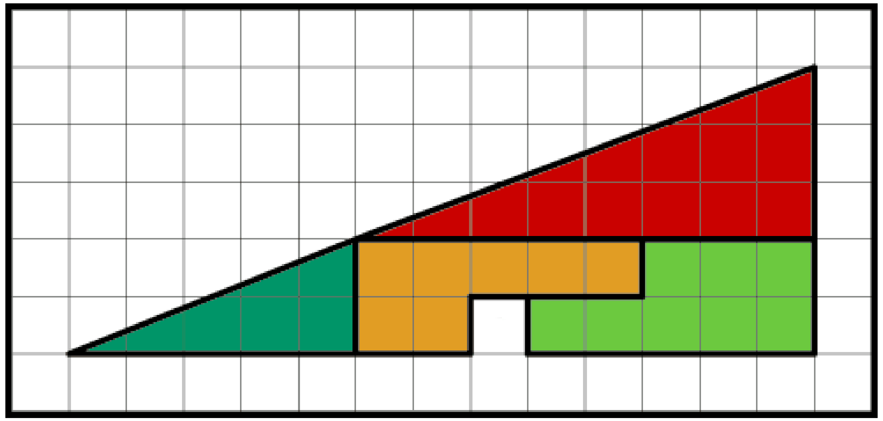
Questions?

Assignment 6

Logical warmup

- Consider the following shape to the right:
- Now, consider the next shape, made up of pieces of exactly the same size:
- We have created space out of nowhere!
- How is this possible?





Final exam

- **Final exam:**
  - **Wednesday, April 29, 2026**
  - **8:00 to 10:00 a.m.**
  - **50% longer than previous exams, but you'll have 100% more time**
- There will be short answer, diagrams, and proofs
- It will be comprehensive but weighted toward the last quarter of the course

Set Theory

Set theory basics

- Defining finite and infinite sets
- Definitions of:
  - Subset
  - Proper subset
  - Set equality
- Set operations:
  - Union
  - Intersection
  - Difference
  - Complement
- The empty set
- Partitions
- Cartesian product

Set theory proofs

- Proving a subset relation
  - Element method: Assume an element is in one set and show that it must be in the other set
  - Algebraic laws of set theory: Using the algebraic laws of set theory (given on the next slide), we can show that two sets are equal
- Disproving a universal statement requires a counterexample with specific sets

Laws of set theory

Name	Law	Dual
Commutative	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associative	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributive	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity	$A \cup \emptyset = A$	$A \cap U = A$
Complement	$A \cup A^c = U$	$A \cap A^c = \emptyset$
Double Complement	$(A^c)^c = A$	
Idempotent	$A \cup A = A$	$A \cap A = A$
Universal Bound	$A \cup U = U$	$A \cap \emptyset = \emptyset$
De Morgan's	$(A \cup B)^c = A^c \cap B^c$	$(A \cap B)^c = A^c \cup B^c$
Absorption	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
Complements of U and $\emptyset$	$U^c = \emptyset$	$\emptyset^c = U$
Set Difference	$A - B = A \cap B^c$	

Set theory proof example

- Use the element method to prove the following:
- For all sets A, B, and C, if $A \subseteq B$ then $A \cap C \subseteq B \cap C$

Russell's paradox

- It is possible to give a description for a set which describes a set that does not actually exist
- For a well-defined set, we should be able to say whether or not a given element is or is not a member
- If we can find an element that must be in a specific set and must not be in a specific set, that set is not well defined
- **Watch out for definitions that are logically inconsistent!**

Functions

- **One-to-one** (**injective**) functions
- **Onto** (**surjective**) functions
- If a function is both one-to-one and onto, we call it **bijective**

Cardinality

- Cardinality is the number of things in a set
  - It is reflexive, symmetric, and transitive
- Two sets have the same cardinality if a bijective function maps every element in one to an element in the other
- Any set with the same cardinality as positive integers is called **countably infinite**

Relations

- **Relations** are generalizations of functions
- In a function, an element of the domain must map to exactly one element of the co-domain
- In a relation, an element from one set can be related to any number (from zero up to infinity) of other elements
- Like functions, we're usually going to focus on binary relations
- We can define any binary relation between sets A and B as a subset of $A \times B$

Properties

- Relation R is **reflexive** iff for all $x \in A, (x, x) \in R$
  - R is **not** reflexive if there is an $x \in A$, such that $(x, x) \notin R$
- Relation R is **symmetric** iff for all $x, y \in A$, if $(x, y) \in R$ then $(y, x) \in R$
  - R is **not** symmetric if there is an $x, y \in A$, such that $(x, y) \in R$ but $(y, x) \notin R$
- Relation R is **transitive** iff for all $x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$
  - R is **not** transitive if there is an $x, y, z \in A$, such that $(x, y) \in R$ and $(y, z) \in R$ but $(x, z) \notin R$
- Relation R is **antisymmetric** iff for all a and b in A, if $a\ R\ b$ and $b\ R\ a$, then $a = b$
  - If two different elements are related to each other, then the relation is **not** antisymmetric

Kinds of relations

- Any relation R is that is reflexive, symmetric, and transitive induces a partition
  - We call a relation with these three properties an **equivalence relation**
- Any relation R that is reflexive, antisymmetric, and transitive is called a **partial order**

Counting and Probability

Multiplication rule

- If an operation has k steps such that
  - Step 1 can be performed in n_1 ways
  - Step 2 can be performed in n_2 ways

    ...
  - Step k can be performed in n_k ways
- Then, the entire operation can be performed in $\prod_{i=1}^{k} n_i = n_1 n_2 \dots n_k$ ways
- This rule only applies when each step always takes the same number of ways (unlike the previous possibility tree example)

Addition and inclusion/exclusion rules

- If a finite set A equals the union of k distinct mutually disjoint subsets A_1, A_2, … A_k, then:

$$N(A) = N(A_1) + N(A_2) + \ldots + N(A_k)$$

- If A, B, C are any finite sets, then

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$

- And

$$N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C)$$

Handy dandy guide to counting

- This is a quick reminder of all the different ways you can count things:

	Order Matters	Order Doesn't Matter
Repetition Allowed	n^k	$\binom{k+n-1}{k}$
Repetition Not Allowed	$P(n,k)$	$\binom{n}{k}$

How many ways are there to…

- Order the letters in the word "lunacy"
- Order any four letters in the word " lunacy"?
- Order the letters in the word "sassafras"
  - If all letters are considered distinct?
  - If identical letters cannot be told apart?
- Select a soccer team of 11 people from 50 candidates?
- Select a soccer team of 11 people from 50 candidates and make one of the 11 the goalie?
- Select a soccer team of 11 people from 25 men and 25 women where five must be women, five must be men, and the goalie can be either?

How many ways are there to…

- How many Java identifiers of exactly 8 characters are possible?
- How many Java identifiers of up to 8 characters are possible?
- How many ways are there to select a basket of 15 pastries made up of:
  - Croissants
  - Pain au chocolat
  - Kouign-amann
  - Cheese danish

Binomial theorem

- $a + b$ is called a **binomial**
- Using combinations (or Pascal's Triangle) gives an easy way to compute $(a + b)^n$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Probability axioms

- Let A and B be events in the sample space S
  - $0 \leq P(A) \leq 1$
  - $P(\emptyset) = 0$ and $P(S) = 1$
  - If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$
  - It is clear then that $P(A^c) = 1 - P(A)$
  - More generally, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- All these axioms can be derived from set theory and the definition of probability

Expected value

- **Expected value** is one of the most important concepts in probability, especially if you want to gamble
- The expected value is simply the sum of all events, weighted by their probabilities
- If you have n outcomes with real number values $a_1, a_2, a_3, \dots\ a_n$, each of which has probability $p_1, p_2, p_3, \dots\ p_n$, then the expected value is:

$$\sum_{k=1}^{n} a_k p_k$$

Conditional probability

- Given that some event A has happened, the probability that some event B will happen is called conditional probability
- This probability is:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Ticket Out the Door

Upcoming

Next time…

- Review of third third of the course

Reminders

- Finish Assignment 6
  - **Due Wednesday!**
- **Final exam:**
  - **Wednesday, April 29, 2026**
  - **8:00 to 10:00 a.m.**